DIVERGENT FORM OF THE NONLINEAR THERMOELASTICITY EQUATIONS

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The complete system of nonlinear thermoelasticity equations is considered in this paper in the form of a system of energy, momentum, and strain compatibility conservation laws in arbitrary curvilinear moving coordinates. Utilization of the equations in such a form has a number of advantages, related both to the investigation of the most important properties of a medium, and to the construction of conservative numerical methods.

Characteristic properties are studied in an adiabatic approximation, symmetrization is performed, sufficient conditions are formulated for the equations of the dynamics of an arbitrary thermoelastic body to be hyperbolic.

A closed system of relationships on strong discontinuities is examined. Conditions for the solvability of the problem determining the quantities behind the shock front are clarified for a known state before the front and a given wave velocity.

1. Kinematics

Let us consider finite deformations of a nonlinear thermoelastic anisotropic medium. Let $\mathbf{X} = X^i\mathbf{э}_i$ be the radius-vector of a material particle of the body in the initial, reference configuration, $\mathbf{x} = x^i\mathbf{э}_i$ in the running, actual configuration, $\mathbf{э}_i$ the orthonormal basis of an Eulerian spatial coordinate system, t the time, $v^i = (\partial x^i/\partial t)\big|_{X^m}$, the particle velocity, and $F^i_{\cdot j} = \partial x^i/\partial X^j$, the gradient of the strain (distortion). Assuming mutual one-to-oneness of the mapping

$$x^i = x^i(X^m,\ t),\ i,\ m = 1,\ 2,\ 3,$$

we obtain a relation between the gradient of the deformation and the velocity of the particle

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \nabla\mathbf{v},\ \frac{\partial F^i_j}{\partial t}\bigg|_{x^m} + v^k\frac{\partial F^i_j}{\partial x^k} = \frac{\partial v^i}{\partial x^k}F^k_j, \tag{1.1}$$

which is well known in the mechanics of a continuous medium [1, 2].

Equation (1.1) can be represented as a differential conservation law. In the variables (x^i, t) the divergent form (1.1) is written in the form

$$\frac{\partial\left(\frac{1}{\Delta}F^i_j\right)}{\partial t}\Bigg|_{x^m} + \frac{\partial}{\partial x^k}\left\{\frac{1}{\Delta}\left(v^kF^i_j - v^iF^k_j\right)\right\} = 0,\ \Delta = \det\mathbf{F}. \tag{1.2}$$

The equivalence of (1.1) and (1.2) can be shown if the formulas

$$\frac{\partial\Delta}{\partial t}\bigg|_{x^m} = \Delta\frac{\partial v^k}{\partial x^k},\ \frac{\partial\Delta}{\partial x^k} = \Delta F_a^{-1b}\frac{\partial F^a_b}{\partial x^k},\ \frac{\partial}{\partial x^k}\left(\frac{1}{\Delta}F^k_j\right) = 0 \tag{1.3}$$

are used. To prove the first two formulas of (1.3), we note that $\Delta = e_{ijm}F^i_1F^j_2F^m_3$, where e_{ijm} is the unit antisymmetric tensor, and $\partial\Delta/\partial F^a_b = \Delta F_a^{-1b}$. The validity of the third relationship in (1.3) follows from the fact that

$$\partial F^k_j/\partial x^k = F_k^{-1m}\left(\partial F^k_m/\partial x^a\right)F^a_j.$$

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Now if we use the mass conservation law $\rho\Delta = \rho_0$, where ρ_0, ρ is the density of the medium in the reference and actual configurations, then (1.2) can be written for the case ρ_0 = const in the form

$$\partial(\rho F_j^i)/\partial t + \frac{\partial}{\partial x^k}\{\rho v^k F_j^i - \rho v^i F_j^k\} = 0. \tag{1.4}$$

A direct consequence of the relations $\rho\Delta = \rho_0$ and $\dot{\Delta} = \Delta \partial v^k/\partial x^k$ is the continuity equation

$$\partial\rho/\partial t + \partial(\rho v^k)/\partial x^k = 0; \tag{1.5}$$

however, we shall not later use it as an independent equation. All the information contained in (1.5) in the domains of smooth solutions and in the integral equalities for domains with discontinuous solutions is already included in the differential equation (1.4) and the finite algebraic relation $\rho = \rho(\mathbf{F}) = \rho_0/\det \mathbf{F}$.

It should also be noted that (1.4) is invariant relative to replacement of the reference system (with respect to imposition of the motion as a rigid whole).

2. Thermodynamics and Governing Equations

Let U be the density of the internal energy; $E = U + v_i v^i/2$, density of the total energy; $\mathbf{q} = q^i \mathbf{э}_i$, heat flux vector through unit area per unit time; r, specific heat liberation; $\mathbf{b} = b^i \mathbf{э}_i$, mass force vector; and $\sigma = \sigma^{ij}\mathbf{э}_i\mathbf{э}_j$, Cauchy stress tensor. Considering only the thermal and mechanical interaction, the total energy conservation law (first principle of thermodynamics) and the momentum and moment of momentum conservation laws for a nonpolar medium in an inertial coordinate system can be written in smooth solution domains in the form [1]

$$\rho dv^i/dt = \partial\sigma^{ik}/\partial x^k + \rho b^i, \quad \sigma^{ik} = \sigma^{ki}; \tag{2.1}$$

$$\rho dU/dt = \sigma_i^k \partial v^i/\partial x^k + \partial q^k/\partial x^k + \rho r. \tag{2.2}$$

Let us moreover use the second principle of thermodynamics whose differential form is written for smooth functions in the form [2]

$$\rho\frac{d\eta}{dt} - \frac{\partial}{\partial x^k}\left(\frac{1}{\Theta}q^k\right) - \frac{1}{\Theta}\rho r \geqslant 0, \tag{2.3}$$

where $\Theta > 0$ is the temperature; η is the specific entropy. Introducing the free energy $A = U - \Theta\eta$ and using the expression for dU/dt from (2.2) then (2.3) can be transformed to the form

$$-\rho\frac{dA}{dt} - \rho\eta\frac{d\Theta}{dt} + \sigma_i^k\frac{\partial v^i}{\partial x^k} + \frac{1}{\Theta}q^k\frac{\partial\Theta}{\partial x^k} \geqslant 0. \tag{2.4}$$

For a nonlinear elastic medium, postulating that

$$\begin{gathered} A = A(F_n^m, \Theta, \gamma_m), \quad \sigma_{ij} = \sigma_{ij}(F_n^m, \Theta, \gamma_m), \\ \eta = \eta(F_n^m, \Theta, \gamma_m), \quad q^i = q^i(F_n^m, \Theta, \gamma_m), \end{gathered} \tag{2.5}$$

where $\gamma_m = \partial\Theta/\partial x^m$ and A, σ_{ij}, η, q^i are sufficiently smooth functions of their arguments, and using (2.4) we obtain [2]

$$\partial A/\partial\gamma_m = 0, \quad \eta = -\partial A/\partial\Theta, \quad \sigma_i^k = \rho F_m^k \partial A/\partial F_m^i, \quad q^m\gamma_m \geqslant 0. \tag{2.6}$$

Let us note that the symmetry condition $\sigma_{ij} = \sigma_{ji}$ imposes the following constraint on the form of the admissible free-energy functions:

$$(\partial A/\partial F_j^i)(F_j^k g^{mi} - F_j^m g^{ki}) = 0, \tag{2.7}$$

where g^{mi} is the metric tensor of the coordinate system x^i. For isotropic bodies the relationships (2.7) are satisfied automatically. In the case of arbitrary anisotropy, (2.7) is three equations which should be satisfied by any nonlinearly elastic membrane medium.

Equation (2.2) for the internal energy can be written with (2.6) taken into account, in the form

$$\rho c_F\frac{d\Theta}{dt} = \Theta\frac{\partial\sigma_i^k}{\partial\Theta}\frac{\partial v^i}{\partial x_s^k} + \frac{\partial q^k}{\partial x^k} + \rho r, \tag{2.8}$$

where $c_F = \Theta\partial\eta/\partial\Theta = -\Theta\,\partial^2 A/\partial\Theta^2 = \partial U/\partial\Theta$ is the specific heat of the medium under constant deformation.

3. Complete System of Equations

Therefore, the complete system of equations of nonlinear thermoelasticity is written in the form of nondivergent differential equations (1.1), (2.1), and (2.8) in the variables (x^i, t)

$$\rho\frac{dv^i}{dt}=\frac{\partial\sigma^{ik}}{\partial F_n^m}\frac{\partial F_n^m}{\partial x^k}+\frac{\partial\sigma^{ik}}{\partial\Theta}\frac{\partial\Theta}{\partial x^k}+\rho b^i,$$
$$\rho c_F\frac{d\Theta}{dt}=\Theta\frac{\partial\sigma_i^k}{\partial\Theta}\frac{\partial v^i}{\partial x^k}+\frac{\partial q^k}{\partial x^k}+\rho r,\quad \frac{dF_j^i}{dt}=F_j^k\frac{\partial v^i}{\partial x^k} \tag{3.1}$$

and the finite relationships

$$A=A\left(F_n^m,\Theta\right),\ c_F=-\Theta\partial^2A/\partial\Theta^2,\ \sigma_j^i=\rho F_k^i\partial A/\partial F_k^j,$$
$$q^k=q^k\left(F_n^m,\Theta,\partial\Theta/\partial x^m\right),\ \rho=\rho_0/\det\|F_j^i\|,\ q^k\partial\Theta/\partial x^k\geqslant 0.$$

Equations (3.1) can be formulated in the form of a system of differential energy and momentum conservation laws [1, 2] and a strain compatibility conservation law expressed by the relationship (1.4):

$$\frac{\partial(\rho E)}{\partial t}+\frac{\partial\left(\rho Ev^k-\sigma^{ik}v_i-q^k\right)}{\partial x^k}=\rho\left(r+b^iv_i\right), \tag{3.2}$$
$$\frac{\partial(\rho v^i)}{\partial t}+\frac{\partial\left(\rho v^iv^k-\sigma^{ik}\right)}{\partial x^k}=\rho b^i,\ \frac{\partial\left(\rho F_j^i\right)}{\partial t}+\frac{\partial\left(\rho v^kF_j^i-\rho v^iF_j^k\right)}{\partial x^k}=0.$$

The system of equations (3.2) possesses a number of advantages over the traditional formulation of the equations of nonlinear elasticity theory [1, 2]. Writing the whole system in divergent form permits determination of not only the classical solution in domains of sufficient smoothness, but also the generalized, weak solution [3] in domains including discontinuities. All the relationships on discontinuities known in nonlinear elasticity theory follow from (3.2) by a standard method [3]. Finally, the system (3.2) is convenient for the construction of conservative difference methods of numerical solution of dynamic and static problems, methods in which the integral conservation laws are satisfied exactly, and there is a theoretical foundation of the "through" computation [4].

However, utilization of the Euler variables (x^i, t) is often fraught with definite difficulties, especially in numerical investigations. The new approach, developed in recent years in a number of papers [5, 6], is the method of moving coordinates. The method consists of introducing moving coordinate grids, which differ from Lagrangian in the general case, whose lines coincide with the extracted singularities of the problem (boundaries, contact discontinuities, shocks, etc.) and satisfy a number of additional requirements.

Let

$$\eta^i=\eta^i(x^k,t),\ \det\|\partial\eta^i/\partial x^m\|\neq 0 \tag{3.3}$$

be a mutually one-to-one, twice continuously differentiable transformation of the space (x^k, t) into (η^i, t) chosen from some considerations. Let us use the notation

$$\eta_k^i=\frac{\partial\eta^i}{\partial x^k},\ x_j^k=\frac{\partial x^k}{\partial\eta^j},\ \widetilde{\Delta}=\det\|x_k^i\|,\ w^i=\left.\frac{\partial x^i}{\partial t}\right|_{\eta^m}.$$

Then the divergent equation in the variables (x^i, t)

$$\left.\frac{\partial A_{i_1i_2\ldots i_N}^{j_1j_2\ldots j_R}}{\partial t}\right|_{x^n}+\frac{\partial B_{i_1i_2\ldots i_N}^{kj_1j_2\ldots j_R}}{\partial x^k}=f_{i_1i_2\ldots i_N}^{j_1j_2\ldots j_R},\ R\geqslant 0,\ N\geqslant 0,\ i_m,\ j_m=1,2,3, \tag{3.4}$$

relating the N-covariant and R-contravariant tensor A with the N-covariant and R + 1-contravariant tensor B is equivalent to the divergent equation in the variables (η^m, t)

$$\left.\frac{\partial\left(\widetilde{\Delta}A_{i_1i_2\ldots i_N}^{j_1j_2\ldots j_R}\right)}{\partial t}\right|_{\eta^n}+\frac{\partial\left(\widetilde{\Delta}\widetilde{B}_{i_1i_2\ldots i_N}^{mj_1j_2\ldots j_R}\right)}{\partial\eta^m}=\widetilde{\Delta}f_{i_1i_2\ldots i_N}^{j_1j_2\ldots j_R}, \tag{3.5}$$

where

$$\widetilde{B}_{i_1i_2\ldots i_N}^{mj_1j_2\ldots j_R}=\eta_k^m\left(B_{i_1i_2\ldots i_N}^{kj_1j_2\ldots j_R}-w^kA_{i_1i_2\ldots i_N}^{j_1j_2\ldots j_R}\right). \tag{3.6}$$

The validity of the assertion follows from the chain of manipulations

$$\left.\frac{\partial A}{\partial t}\right|_{x^k}+\frac{\partial B^m}{\partial x^m}=\left.\frac{\partial A}{\partial t}\right|_{x^k}+\frac{\partial}{\partial x^m}(Aw^m+\tilde{B}^k x_k^m)=\left.\frac{\partial A}{\partial t}\right|_{x^k}+w^m\frac{\partial A}{\partial x^m}+\frac{\partial w^m}{\partial x^m}A$$

$$+\frac{\partial \tilde{B}^k}{\partial x^m}x_k^m+\tilde{B}^k\frac{\partial x_k^m}{\partial x^m}=\left.\frac{\partial A}{\partial t}\right|_{\eta^i}+\frac{\partial \tilde{B}^k}{\partial \eta^k}+A\frac{\partial w^k}{\partial x^k}+\tilde{B}^k\frac{\partial x_k^m}{\partial x^m}$$

and the relations

$$\frac{\partial x_k^a}{\partial x^a}=\frac{1}{\tilde{\Delta}}\frac{\partial \tilde{\Delta}}{\partial \eta^k},\quad \frac{\partial w^a}{\partial x^a}=\left.\frac{1}{\tilde{\Delta}}\frac{\partial \tilde{\Delta}}{\partial t}\right|_{\eta^m},$$

which are derived analogously to (1.3).

Now, the system of nonlinear thermoelasticity conservation laws is written with (3.4)-(3.6) taken into account, in the curvilinear coordinates (η^i, t) in the form

$$\begin{gathered}\frac{\partial(\rho\tilde{\Delta}E)}{\partial t}+\frac{\partial}{\partial \eta^m}\{\tilde{\Delta}[\rho E(\tilde{v}^m-\tilde{w}^m)-\eta_k^m\sigma^{ki}v_i-\eta_k^m q^k]\}=\tilde{\Delta}\rho(r+b^i v_i),\\ \frac{\partial(\rho\tilde{\Delta}v^i)}{\partial t}+\frac{\partial}{\partial \eta^m}\{\tilde{\Delta}[\rho v^i(\tilde{v}^m-\tilde{w}^m)-\eta_k^m\sigma^{ki}]\}=\tilde{\Delta}\rho b^i,\\ \frac{\partial(\rho\tilde{\Delta}F_j^i)}{\partial t}+\frac{\partial}{\partial \eta^m}\{\tilde{\Delta}\rho[F_j^i(\tilde{v}^m-\tilde{w}^m)-v^i\eta_k^m F_j^k]\}=0,\end{gathered}\tag{3.7}$$

where $\tilde{v}^m=\eta_k^m v^k$; $\tilde{w}^m=\eta_k^m w^k$.

Let us note that because of using the tensor and vector components referred to the coordinates x^i in (3.7), although the equations are themselves written in (η^m, t) the appearance of additional "sources" not due to the physical content in the right side is avoided.

Here (3.7) takes on a quite simple form in the case of Lagrange coordinates X^m when $w^m \equiv v^m$, $\tilde{\Delta}\rho=\rho_0$, $\eta_k^m=F_k^{-1m}$:

$$\begin{gathered}\frac{\partial E}{\partial t}-\frac{\partial}{\partial X^m}\left[\frac{1}{\rho}F_k^{-1m}(\sigma^{ki}v_i+q^k)\right]=r+b^i v_i,\\ \frac{\partial v^i}{\partial t}-\frac{\partial}{\partial X^m}\left(\frac{1}{\rho}F_k^{-1m}\sigma^{ki}\right)=b^i,\ \frac{\partial F_j^i}{\partial t}-\frac{\partial}{\partial X^m}(\delta_j^m v^i)=0.\end{gathered}\tag{3.8}$$

4. Characteristics

Now, let us consider the characteristic properties of the system (3.1) in an adiabatic approximation ($q^i \equiv 0$). In this case (3.1) is a quasilinear system

$$\frac{du_\alpha}{dt}+A_{\alpha\beta}^k(u_\gamma)\frac{\partial u_\beta}{\partial x^k}=f_\alpha,\ \alpha,\beta,\gamma=1,\ 2,\ldots,13,\ k=1,2,3,$$

where $u_\alpha=\{\Theta, v^i, F_j^i\}$; $f_\alpha=\{r/c_F,\ b^i,\ 0\}$. Let $\varphi(t, x^i)=0$ be the equation of the characteristic surface, $D=-\frac{\partial\varphi}{\partial t}\Big/\left\{\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^m}\right\}^{1/2}$, $n_i=\frac{\partial\varphi}{\partial x^i}\Big/\left\{\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^m}\right\}^{1/2}$, the propagation velocity and the normal to this surface. Then the equation of the characteristics has the form [3]

$$\det\|-c\mathbf{I}+\mathbf{A}^k n_k\|=0,\tag{4.1}$$

where $c=D-v^k n_k$ is the velocity of the surface relative to the particles of the medium, and **I** is the unit matrix. Let p_α denote the right eigenvector of the matrix $\mathbf{A}^k n_k$, which it is convenient to represent for the subsequent computations as a set of components $p_\alpha=\{\alpha, \beta_i, \gamma_j^i\}$ corresponding to jumps in the normal derivatives of the temperature, the velocity, and the gradient of the deformation. The equation $(-c\mathbf{I}=\mathbf{A}^k n_k)p=0$ is then written in the form

$$\begin{gathered}\rho c_F c\alpha+\Theta\frac{\partial\sigma_i^k}{\partial\Theta}n_k\beta^i=0,\\ \rho c\beta^i+\frac{\partial\sigma^{ik}}{\partial F_n^m}n_k\gamma_n^m+\frac{\partial\sigma^{ik}}{\partial\Theta}n_k\alpha=0,\ c\gamma_j^i+F_j^k n_k\beta^i=0.\end{gathered}\tag{4.2}$$

In case $c \neq 0$, then by expressing the quantities α and γ_j^i in terms of β^i from the first and third equations of (4.2), and substituting into the second relationship, we arrive at a system of three equations with a symmetric coefficient matrix

$$\left\{c^2 g_{ij} - (F^a_m n_a)\left(\frac{\partial^2 A}{\partial F^i_m \partial F^j_n} + \frac{\Theta}{c_F}\frac{\partial \eta}{\partial F^i_m}\frac{\partial \eta}{\partial F^i_n}\right)(F^b_n n_b)\right\}\beta^i = 0. \tag{4.3}$$

A condition for the existence of real nonzero values of the propagation velocities of the characteristic surfaces is positive-definiteness of the symmetric quadratic form

$$(F^a_m n_a)\left(\frac{\partial^2 A}{\partial F^i_m \partial F^j_n} + \frac{\Theta}{c_F}\frac{\partial \eta}{\partial F^i_m}\frac{\partial \eta}{\partial F^j_n}\right)(F^b_n n_b)\,\lambda^i\lambda^j > 0$$

for arbitrary $\lambda^i \neq 0$. If the internal energy function $U = U(F^m_n, \eta)$ is used, then this condition can be written in the form

$$(F^a_m n_a)\frac{\partial^2 U}{\partial F^i_m \partial F^j_n}(F^b_n n_b)\,\lambda^i\lambda^j > 0. \tag{4.4}$$

It is also seen from the system (4.2) that $c = 0$ is a multiple root of the characteristic equation (4.1), where $\beta^i = 0$ in this case, and the relation

$$M^{ik}_n \gamma^n_k + m^i\alpha = 0,\; M^{ik}_n = n_s\frac{\partial \sigma^{is}}{\partial F^n_k},\quad m^i = n_s\frac{\partial \sigma^{is}}{\partial \Theta} \tag{4.5}$$

is imposed on γ^m_n and α. It is not possible to make any assertion about the existence of a solution of the homogeneous equation (4.5) without knowledge of the coefficient matrix.

Let us formulate sufficient conditions for which the system of equations (3.1) will be known to be hyperbolic. To do this, we reduce the system to symmetric form by using the nondegenerate replacement of the vector of the solution. Symmetrization and the sufficient conditions for hyperbolicity of the nonlinear gasdynamics and the linear elasticity theory equations with small deformations were examined in [7]. Utilization of the divergent equation (1.4) for the strain gradient F^i_j permitted extension of the solution of this problem to the case of an arbitrary nonlinearly elastic body with finite strains also.

Analogously to [7], we use the additional entropy conservation law for this:

$$\left.\frac{\partial \eta}{\partial t}\right|_{X^m} = \frac{1}{\Theta}r, \tag{4.6}$$

which is valid in an adiabatic approximation in the region of smooth solutions. We now obtain (4.6) as a result of the divergent equations (3.8) written in the Lagrange coordinates X^m. To do this, we multiply the equation of (3.8) by the factor q_0, the second by q_i, the third by q^j_i, where q_0, q_i, q^j_i are as yet unknown functions of Θ, v^i, and F^i_j. Adding the equations and equating to (4.6), we obtain a system for q_0, q_i, q^j_i:

$$\begin{gathered}\frac{\partial \eta}{\partial t} = q_0\frac{\partial E}{\partial t} + q_i\frac{\partial v^i}{\partial t} + q^j_i\frac{\partial F^i_j}{\partial t},\; q_0(r + b^i v_i) + q_i b^i = \frac{1}{\Theta}r,\\ q_0\frac{\partial}{\partial X^m}\left(\frac{1}{\rho}F_k^{-1m}\sigma^{ki}v_i\right) + q_i\frac{\partial}{\partial X^m}\left(\frac{1}{\rho}F_k^{-1m}\sigma^{ki}\right) + q^j_i\frac{\partial}{\partial X^m}(\delta^m_j v^i) = 0,\end{gathered} \tag{4.7}$$

from which

$$q_0 = \frac{1}{\Theta},\; q_i = -\frac{v_i}{\Theta},\; q^j_i = -\frac{1}{\rho\Theta}F_k^{-1j}\sigma^k_i. \tag{4.8}$$

We here assume that the free energy $A = A(F^m_n, \Theta)$ is such that the quantities $\{q_0, q_i, q^j_i\}$ are mutually solvable one-to-one for $\{\Theta, v^i, F^i_j\}$. As will be seen from the sequel, the sufficient conditions for hyperbolicity assures the validity of this assumption.

Relationships (4.7) will be known to be satisfied if the stronger relations

$$\begin{gathered}d\eta = q_0 dE + q_i dv^i + q^j_i dF^i_j,\\ q_0 d\left(\frac{1}{\rho}F_k^{-1m}\sigma^{ki}v_i\right) + q_i d\left(\frac{1}{\rho}F_k^{-1m}\sigma^{ki}\right) + q^j_i d(\delta^m_j v^i) = 0\end{gathered} \tag{4.9}$$

are satisfied, which actually hold with (4.8) and (2.6) taken into account.

Rewriting (4.9) in the form

$$d(q_0E + q_iv^i + q_i^jF_j^i - \eta) = Edq_0 + v^idq_i + F_j^idq_i^j,$$

$$d\left(\frac{1}{\rho} q_0F_k^{-1m}\sigma^{ki}v_i + \frac{1}{\rho} q_iF_k^{-1m}\sigma^{ki} + q_i^j\delta_j^m v^i\right) = \frac{1}{\rho} F_k^{-1m}\sigma^{ki}v_idq_0 + \frac{1}{\rho} F_k^{-1m}\sigma^{ki}dq_i + \delta_j^m v^idq_i^j$$

and using the notation

$$L^0 = \eta - q_0E - q_iv^i - q_i^jF_j^i = \frac{1}{\Theta}\left(\frac{1}{\rho}\sigma_k^k + \frac{1}{2}v_iv^i - A\right), \tag{4.10}$$

$$L^m = \frac{1}{\rho} q_0F_k^{-1m}\sigma^{ki}v_i + \frac{1}{\rho} q_iF_k^{-1m}\sigma^{ki} + q_i^j\delta_j^m v^i = -\frac{1}{\rho\Theta} F_k^{-1m}\sigma^{ki}v_i,$$

we find

$$E = -\frac{\partial L^0}{\partial q_0},\ v^i = -\frac{\partial L^0}{\partial q_i},\ F_j^i = -\frac{\partial L^0}{\partial q_i^j}, \tag{4.11}$$

$$\frac{1}{\rho} F_k^{-1m}\sigma^{ki}v_i = \frac{\partial L^m}{\partial q_0},\ \frac{1}{\rho} F_k^{-1m}\sigma^{ki} = \frac{\partial L^m}{\partial q_i},\ \delta_j^m v^i = \frac{\partial L^m}{\partial q_i^j}.$$

Therefore, taking (4.11) into account, the system of differential equations of nonlinear elasticity theory can be written in the form of a symmetric first order system for four generating functions L^0, L^m ($m = 1, 2, 3$) defined by (4.10):

$$\frac{\partial L_{q_\alpha}^0}{\partial t} + \frac{\partial L_{q_\alpha}^m}{\partial X^m} = L_{q_\alpha q_\beta}^0 \frac{\partial q_\beta}{\partial t} + L_{q_\alpha q_\beta}^m \frac{\partial q_\beta}{\partial X^m} = f_\alpha, \tag{4.12}$$

where

$$L_{q_\alpha}^{0,m} = \partial L^{0,m}/\partial q_\alpha;\ L_{q_\alpha q_\beta}^{0,m} = \partial^2 L^{0,m}/\partial q_\alpha \partial q_\beta;$$

$$q_\alpha = \{q_0, q_i, q_i^j\};\ f_\alpha = \{-(r + b^iv_i), -b^i, 0\}.$$

In order for system (4.12) to be hyperbolic, it is sufficient that the matrix $L^0_{q_\alpha q_\beta}$ be positive-definite. In this case there exists a nondegenerate transformation that simultaneously reduces the matrices $L^0_{q_\alpha q_\beta}$ and $L^m_{q_\alpha q_\beta}n_m$ in the characteristic equation

$$\det\left\| -cL_{q_\alpha q_\beta}^0 + L_{q_\alpha q_\beta}^m n_m \right\| = 0$$

to diagonal form.

The positive definiteness of $L^0_{q_\alpha q_\beta}$ is equivalent to convexity of the function $L^0 = L^0(1/\Theta, -v_i/\Theta, -F_k^{-1j}\sigma_i^k/(\rho\Theta))$ relative to all its arguments. Let us use the property [7] by virtue of which the function M is such that

$$M(1/q_0, q_1/q_0, \ldots, q_n/q_0) = (1/q_0)L^0(q_0, q_1, \ldots, q_n)$$

is convex if L^0 is convex. The sufficient condition for hyperbolicity can be formulated in the form of the condition for convexity of the function $M(\Theta, v_i, F_k^{-1j}\sigma_i^k/\rho) = \sigma_k^k/\rho + v_iv^i/2 - A$. By constructing the Legendre transform H of the function M, we arrive at the following result:

$$H(\eta, v_i, F_j^i) = U(F_j^i, \eta) + \frac{1}{2}v_iv^i.$$

Therefore, the matrix $L^0_{q_\alpha q_\beta}$ will be positive-definite if the quadratic form

$$(\partial^2 U/\partial g_\alpha \partial g_\beta)\lambda^\alpha\lambda^\beta > 0 \tag{4.13}$$

is positive-definite, where $g_\alpha = \{F_n^m, \eta\}$; $\lambda^\alpha \neq 0$ is an arbitrary vector. Condition (4.13) is known to be stronger than the condition (4.4).

Let us note that because of (4.11) and the convexity of L^0 the determinant is $\partial(E, v^i, F_j^i)/\partial(q_0, q_i, q_i^j)$ which assures the one-to-one mapping $(\Theta, v^i, F_j^i) \leftrightarrow (q_0, q_i, q_i^j)$ for $c_F \neq 0$, that was assumed above.

5. Strong Discontinuities

For the system of divergent equations (3.2) or (3.7) a weak or generalized solution [3] can be determined that is valid not only in the smoothness domains, but also in the presence of surfaces of discontinuity. Let $\varphi(x^i, t) = 0$ be the equation of such a surface. Let D denote the normal component of the motion velocity, and n_i the vector normal to the surface under consideration. Then the relations on a strong discontinuity follow from (3.2):

$$[\rho GE] + [\sigma^{ik}v_i + q^k]n_k = 0, \quad [\rho Gv^i] + [\sigma^{ik}]n_k = 0, \qquad (5.1)$$
$$[\rho GF^i_j] + [\rho v^i F^k_j]\, n_k = 0,$$

where $G = D - v^i n_i$ is the velocity of the discontinuity motion relative to particles of the medium, and $[a]$ is the jump in the quantity a on the discontinuity.

Combining the equation for the jump F^i_j with n_i, we obtain for nonstationary discontinuities ($D = 0$)

$$[\rho F^k_j n_k] = \left[\frac{\rho_0}{\det\|F^m_n\|}\, F^k_j n_k\right] = 0. \qquad (5.2)$$

The relationship (5.2) expresses the continuity of the normal to the surface of discontinuity in the reference configuration if the density of the latter is constant.

Taking account of (5.2), the continuity of the mass flow follows from (5.1):

$$[\rho G] = 0, \qquad (5.3)$$

ordinarily obtainable from the continuity equation [1]. To show this, we identify a point on the surface of discontinuity by using the curvilinear coordinates ξ^α ($\alpha = 1, 2$). Let $x^i = x^i(X^m(\xi^\alpha, t), t)$ be the radius-vector of some fixed point ξ^α. Then

$$D^i = \frac{\partial x^i}{\partial t}\bigg|_{\xi^\alpha} = \frac{\partial x^i}{\partial t}\bigg|_{X^m} + \frac{\partial x^i}{\partial X^a}\frac{\partial X^a}{\partial t}\bigg|_{\xi^\alpha} = v^i + F^i_a D^a_*,$$

where $D^a_* = (\partial X^a/\partial t)\big|_{\xi^\alpha}$ is the velocity of the surface relative to the reference configuration. Hence

$$[\rho G] = [\rho\,(D^i - v^i)\, n_i] = [\rho F^k_i n_k D^i_*] = [\rho F^k_i n_k]\, D^i_* = 0.$$

The relationships (5.2) and (5.3) permit writing the equation for the jump in F^i_j in the form

$$[F^i_j] = h^i \rho F^k_j n_k,\ h^i = -\frac{1}{\rho G}[v^i], \qquad (5.4)$$

after which the first two equations in (5.1) take the form

$$\rho G\left\{[U] - \frac{1}{2}(\sigma^k_i + \overset{\circ}{\sigma}{}^k_i)\, h^i n_k\right\} + [q^k]\, n_k = 0, \qquad (5.5)$$
$$[\sigma^{ik}]n_k - (\rho G)^2 h^i = 0,$$

where $\overset{\circ}{\sigma}{}^k_i$, σ^k_i is the stress tensor ahead of and behind the shock front.

Now, let us examine the following problem in an adiabatic approximation ($q^i \equiv 0$). Let the state of a medium be known, i.e., the quantities $\overset{\circ}{F}{}^m_n$, $\overset{\circ}{\eta}$ ahead of a shock front with a given velocity of motion G. Find the necessary conditions for which the state of the medium behind the front, given by the quantities

$$F^i_j = \overset{\circ}{F}{}^i_j + h^i \rho \overset{\circ}{F}{}^k_j n_k,\ \eta = \overset{\circ}{\eta} + \tau,\ \ \tau = [\eta],$$

is determined in a unique manner.

Let us consider the relationship (5.5) as a system of equation in the unknowns h^m and τ

$$\Phi^0(h^m, \tau) = U(\overset{\circ}{F}{}^m_n + h^m\rho\overset{\circ}{F}{}^k_n n_k,\ \overset{\circ}{\eta} + \tau) - U(\overset{\circ}{F}{}^m_n\ \overset{\circ}{\eta}) - \frac{1}{2} n_k h^i \{\sigma^k_i(\overset{\circ}{F}{}^m_n + h^m\rho\overset{\circ}{F}{}^a_n n_a,\ \overset{\circ}{\eta} + \tau) + \sigma^k_i(\overset{\circ}{F}{}^m_n \overset{\circ}{\eta})\} = 0, \qquad (5.6)$$
$$\Phi^i(h^m, \tau) = n_k\{\sigma^{ki}(\overset{\circ}{F}{}^m_n + h^m\rho\overset{\circ}{F}{}^a_n n_a,\ \overset{\circ}{\eta} + \tau) - \sigma^{ki}(\overset{\circ}{F}{}^m_n,\ \overset{\circ}{\eta})\} - (\rho G)^2 h^i = 0.$$

In conformity with the existence theorem for implicit functions, the necessary condition for solvability of the system (5.6) for h^i and τ is

$$\partial(\Phi^1, \Phi^2, \Phi^3, \Phi^0)/\partial(h^1, h^2, h^3, \tau) \neq 0, \tag{5.7}$$

where Φ^0, Φ^i are sufficiently smooth functions of their arguments.

Evaluating the derivatives in (5.7), and utilizing the relationship

$$\frac{\partial\Phi^0}{\partial h^j} + \frac{1}{2} h_i \frac{\partial\Phi^i}{\partial h^j} = 0,$$

the inequality (5.7) can be written in the form

$$\Theta \det \left\| (\rho G)^2 g_{ij} - \left(\rho F^a_m n_a\right) \frac{\partial^2 U}{\partial F^i_m \partial F^j_n} \left(\rho F^b_n n_b\right) \right\| \neq 0. \tag{5.8}$$

Comparing (5.8) and (4.4), we find that the desired condition is $\rho G \neq \rho c$, i.e., the shock velocity should not equal the propagation velocity of the characteristic surface.

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TAKING INTO ACCOUNT THE STRUCTURAL INHOMOGENEITY OF A COMPOSITE MATERIAL IN ESTIMATING ADHESIVE STRENGTH

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One of the basic characteristics of a composite material is its adhesive strength. The experimental determination of this characteristic (in the case of a fiber composite) can be based on a measurement of the load, for which a fiber is pulled out of the matrix.

In order to correctly calculate the adhesive strength from the results of such tests, it is necessary, however, to solve a complex mechanical problem of the distribution of contact stresses between the fiber and the matrix. The use of rigorous methods for analyzing composites at the constituent component level does not permit obtaining at the present time an exact analytic solution of the corresponding problem in the theory of elasticity. For this reason, the engineering approach [1-3], in which it is assumed that the fibers function only under tension, while the matrix only functions under shear, is widely used. Evidently, with this method, it is impossible to take into account the possible singularity of the contact stresses at locations where the fiber and the matrix join on the free boundary. In addition, in using a simplified model, there arises the natural problem of the limits of applicability of the corresponding solutions even outside the regions of concentrated stresses.

A detailed representation of the stressed state can be obtained by the finite-element method. However, in order to apply numerical methods efficiently, preliminary analytic solutions, which correctly reflect the basic

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